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Defect lines and boundary flows

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Abstract: Using the properties of defect lines, we study boundary renormalisation group flows. We find that when there exists a flow between maximally symmetric boundary conditions a and b then there also exists a boundary flow between $c \times a$ and $c \times b$ where $\times$ denotes the fusion product. We also discuss applications of this simple observation.

Keywords: Conformal Field Models in String Theory, Tachyon Condensation, Boundary Quantum Field Theory.

Contents

1. Introduction

Conformal field theories describe statistical systems at criticality. In two dimensions, the classification of classes of conformal field theory has provided an understanding of the notion of universality. When one introduces a boundary to the system, new elements arise and one may attempt to classify boundary critical phenomena.

The machinery of rational conformal field theory is well developed and a great deal is known about the systems at criticality. Most less well understood are theories away from the critical point. A useful way to study such theories is to deform a well known conformal field theory by some relevant perturbation. Using the knowledge of the original theory one may study the perturbed theory using a variety of techniques: Perturbation theory and the truncated conformal space approach being two. If the perturbation is also integrable one can also use a variety of powerful results from the theory of integrable systems: for example, TBA methods or non-linear integrable equations.

In this note, we consider boundary perturbations within rational conformal field theory. Using the technology of defect lines we observe a useful embedding of operator algebras of the boundary condition a into the operator algebra of boundary $a \times b$ for some b (where in the cases we will consider, boundary conditions are labelled by representations of the chiral algebra and $\times$ denotes the fusion product of these representations). This embedding is used to show the main result of this paper.

Assume there exists a boundary flow from boundary condition b to d, then there also exists a flow from boundary condition $a \times b$ to $a \times d$.

Using this result, we prove the existence of many flows conjectured in the literature using the proven existence of some elementary flows. We also consider the conjecture of Fredenhagen and Schomerus [1, 2]. Here we show that a subset of the flows predicted by their conjecture may be obtained from the knowledge of a smaller number of elementary flows.

An outline of the paper is as follows. In section 2 we give a lightning review of boundary rational conformal field theory, to set notation. Here we also review the construction of defect lines following Petkova and Zuber. To study boundary perturbations we will use the formalism of TCSA. Note that we will not do any numerical analysis here, all our results are exact, however the formalism of TCSA provides a powerful tool to study these deformed boundary theories. In section 3 we introduce these ideas. Using all this technology, we prove our theorem in section 4 before demonstrating numerous applications in section 5. We end with some conclusions and directions for further work.

2. A lightning review of rational conformal field theory

We consider a rational conformal field theory (RCFT) built from two copies of a chiral algebra $\mathcal{A}$ containing the Virasoro algebra, and a finite set of representations of $\mathcal{A}$ which we denote by $\mathcal{R}_i$, $i = 1, 2, \ldots, n$. The spectrum of the bulk theory is encoded in the torus partition function,

$$Z = \sum_{i,\bar{i}} Z_{i,\bar{i}}\, \chi_i(q) \bar{\chi}_{\bar{i}}(\bar{q})\,. \tag{2.1}$$

For simplicity, we consider only the charge conjugation modular invariant,

$$Z_{i,\bar{i}} = C_{i,\bar{i}} = \delta_{i,\bar{i}^\vee}\,, \tag{2.2}$$

where C is the charge conjugation matrix relating a representation labelled by i, to that of its conjugate $i^\vee$. In our RCFT we have Verlinde's formula,

$$N_{ij}{}^k = \sum_\ell \frac{S_{i\ell} S_{j\ell} S^*_{k\ell}}{S_{0\ell}}\,, \tag{2.3}$$

where the S-matrix is symmetric, unitary and encodes the modular transformations of the characters,

$$\chi_i(q) = \sum_j S_{ij} \chi_j(\tilde{q})\,, \qquad q = e^{2\pi i \tau}\,, \qquad \tilde{q} = e^{-2\pi i/\tau}\,. \tag{2.4}$$

Boundary conditions preserving a maximal amount of symmetry are given by Cardy's construction, which we now review. Maximally symmetric boundary conditions in RCFT are a restricted class of objects satisfying [3, 4],

$$\left(L_n - \bar{L}_{-n}\right) |a\rangle_\Omega = 0\,, \qquad \left(W_n - (-1)^{s_W} \Omega \bar{W}_{-n}\right) |a\rangle_\Omega = 0\,, \tag{2.5}$$

where L_n are the Virasoro generators, W_n are the modes of a field from the chiral algebra $\mathcal{A}$ and s_W is its spin. We have also introduced a gluing automorphism, Ω [5, 6, 7], although often boundary conditions are only called maximally symmetric if this automorphism is 1.

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These equations may be solved separately in each sector of the bulk Hilbert space where one finds a unique solution, given by an Ishibashi state $|i\rangle\rangle$ with i labelling the sector. The Ishibashi states are a basis of the boundary states but do not correspond to physically realisable boundary conditions. The physically realisable boundary conditions must satisfy further conditions, one of which is called Cardy's condition. To formulate this condition, we consider a cylinder with boundary conditions $\langle a| = \sum_i (\psi_a^i)^* \langle\langle i|$ on the left and $|b\rangle = \sum_i \psi_b^i |i\rangle\rangle$ on the right. With time running around the cylinder, the Hilbert space of the theory is,

$$\mathcal{H}_{ab} = \bigoplus_k n_{ka}{}^b \, \mathcal{R}_k \,, \tag{2.6}$$

The partition function for this system is then,

$$Z_{ab} = \sum_k n_{ka}{}^b \chi_k(q) \,, \tag{2.7}$$

and Cardy's condition is that,

$$\sum_i \psi_a^i (\psi_b^i)^* \, S_{ij} = n_{ja}{}^b \,, \tag{2.8}$$

where S is the modular matrix mentioned above. A general solution to this equation is not known, the difficulty being that the numbers $n_{ja}{}^b$ should be non-negative integers. However, much is understood [8, 9]. One important solution was given by Cardy in his original paper [4] for theories with a charge conjugation modular invariant. Using the Verlinde formula (2.3) one observes that the ansatz,

$$\psi_a^i = \frac{S_{ia}}{\sqrt{S_{i0}}} \tag{2.9}$$

solves Cardy's condition with,

$$n_{jb}{}^a = N_{jb}{}^a \,. \tag{2.10}$$

We will call such boundary conditions, *Cardy boundary conditions*. These are elementary boundary conditions and hence have the property that the identity representation appears only once in the Hilbert space $\mathcal{H}_{aa}$, so that there is a unique operator of conformal weight zero on the boundary and the boundary correlation functions satisfy the physical *clustering property*. More general boundary conditions may be obtained by taking linear combinations of these solutions with positive integer coefficients. Note that Cardy's boundary conditions are also in one to one correspondence with the primary fields of the bulk theory, and so we label them using the same set.

The spectrum of boundary fields is neatly encoded in the cylinder partition function (2.7). A general boundary field carries five labels: $\phi^{(ab)}_{i_\alpha;\mathbf{n}}(x)$, where i labels the representation, a and b label the boundary conditions on the right ($>x$) and left ($<x$) of the insertion point x respectively, while $\alpha = 1, 2, \ldots, N_{ib}{}^a$ accounts for any further multiplicity there may be. The label $\mathbf{n}$ denotes the descendants of the primary field in representation i. The

OPE has the form,

$$x>y\,,\quad \phi^{(ab)}_{i_\alpha;\mathbf{n}}(x)\phi^{(cd)}_{j_\beta;\mathbf{m}}(y)=\sum_{k,\mathbf{p}}\sum_{\gamma=1}^{N_{ij}{}^{k}}\sum_{\delta=1}^{N_{kd}{}^{a}}\delta_{bc}\,C^{(acd)k_\delta}_{i_\alpha j_\beta;\gamma}\;\beta^{\gamma}\!\begin{bmatrix} & k;\mathbf{p} & \\ i;\mathbf{n} & & j;\mathbf{m}\end{bmatrix}(x-y)^{-h_{i;\mathbf{n}}-h_{j;\mathbf{m}}+h_{k;\mathbf{p}}}\phi^{(ad)}_{k_\delta;\mathbf{p}}(y)\,, \tag{2.11}$$

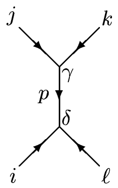
where the numbers $\beta^\gamma[\cdot]$ are entirely determined by the chiral algebra $\mathcal{A}$ and the structure constants C satisfy the sewing constraints of [10]. In the case of the Virasoro minimal models, these sewing constraints have been solved by Runkel [11] who showed that there exists a normalisation in which,

$$C^{(abc)k}_{ij} = \mathsf{F}_{kb}\begin{bmatrix} i & j \\ a & c\end{bmatrix}\,, \tag{2.12}$$

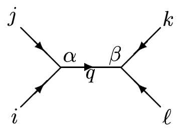
where F is the fusing matrix which encodes the associativity of the operator product expansion. In more general RCFT, this result generalises [9, 12, 13] and there exists a normalisation in which[1] [13],

$$\begin{aligned} C^{(abc)k_\delta}_{i_\alpha j_\beta;\gamma} &= \left(\mathsf{F}_{kb^\vee}\begin{bmatrix} i & j \\ a^\vee & c\end{bmatrix}^{\gamma,\delta}_{\alpha,\beta}\right)^{*}\,, \\ \alpha &= 1,2,\ldots,N_{ib}{}^{a}\,, \qquad \beta = 1,2,\ldots,N_{jc}{}^{b}\,, \\ \gamma &= 1,2,\ldots,N_{ij}{}^{k}\,, \qquad \delta = 1,2,\ldots,N_{kc}{}^{a}\,, \end{aligned} \tag{2.13}$$

where the F-matrix relates different bases for four point conformal blocks (see [14] for a definition),



$$= \sum_{q,\alpha,\beta}\mathsf{F}_{pq}\begin{bmatrix} j & k \\ i & \ell\end{bmatrix}^{\gamma,\delta}_{\alpha,\beta} \tag{2.14}$$



One can check that when one of the legs is replaced by the identity representation, the matrix degenerates and we find,

$$\mathsf{F}_{pq}\begin{bmatrix} j & k \\ i & 0\end{bmatrix}^{\gamma,\delta}_{\alpha,\beta} = \begin{cases}\delta_{pi^\vee}\,\delta_{qk^\vee}\,\delta_{\beta 1}\,\delta_{\delta 1}\,\delta_{\alpha\gamma}\,, & \text{if } N_{ij}{}^{k^\vee}>0\,, \\ 0\,, & \text{otherwise}\,.\end{cases} \tag{2.15}$$

We then have,

$$C^{(ab0)k_\delta}_{i_\alpha j_\beta;\gamma} = \begin{cases}\delta_{ka}\,\delta_{jb}\,\delta_{\beta 1}\,\delta_{\delta 1}\delta_{\alpha\gamma}\,, & \text{if } N_{ib}{}^{a}>0\,, \\ 0\,, & \text{otherwise}\,.\end{cases}\,, \tag{2.16}$$

which we will require later.

[1]We thank Ingo Runkel for discussions on the following.

2.1 Petkova and Zuber's defect lines

The construction of defect lines is very analogous to that of boundary conditions. Following Zuber and Petkova [15], we will define disorder lines as operators X, satisfying the relations,

$$[L_n, X] = [\bar{L}_n, X] = 0\,, \tag{2.17}$$

$$[W_n, X] = [\bar{W}_n, X] = 0\,. \tag{2.18}$$

One may also envisage putting a gluing automorphism into one or both of the second relations. As an operator in the bulk Hilbert space, the operator X is naturally associated to some cycle. For example on the plane, this would be some contractible cycle around the origin. On the cylinder this cycle is non-contractible. The definition (2.18) implies that this operator is invariant under local diffeomorphisms — i.e. the operator X is invariant under distortions of this line. Thus we may associate X to the homotopy class of the cycle to which it is associated.

As in the boundary case, there are also a number of consistency conditions which must be satisfied by the operator X. To formulate the analogue of Cardy's condition, one first notes that as a consequence of (2.17), (2.18), X is a sum of projectors,

$$X = \sum_{j,\bar{\jmath},\alpha,\alpha'} \Psi^{(j,\bar{\jmath};\alpha,\alpha')} P^{(j,\bar{\jmath};\alpha,\alpha')}\,, \tag{2.19}$$

where $\alpha, \alpha' = 1, 2, \ldots, Z_{j,\bar{\jmath}}$ allow for repeated representations in the Hilbert space and if $\{|i, \mathbf{n}\rangle \otimes |\bar{\imath}, \bar{\mathbf{n}}\rangle\}$ denotes an orthonormal basis in $\mathcal{R}_i \otimes \bar{\mathcal{R}}_{\bar{\imath}}$, we write,

$$P^{(i,\bar{\imath};\alpha,\alpha')} = \sum_{\mathbf{n},\bar{\mathbf{n}}} (|i, \mathbf{n}\rangle \otimes |\bar{\imath}, \bar{\mathbf{n}}\rangle)^{(\alpha)} (\langle i, \mathbf{n}| \otimes \langle \bar{\imath}, \bar{\mathbf{n}}|)^{(\alpha')}\,. \tag{2.20}$$

A consistency condition is found by considering a pair of defect lines wrapping a canonical cycle on a torus. Using a hamiltonian picture with "time" moving perpendicular to the lines, the torus partition function may be written,

$$\begin{aligned} Z_{x|y} &\equiv \mathrm{Tr}_{\mathcal{H}} \left(X_x^\dagger X_y \tilde{q}^{L_0 - \frac{c}{24}} \tilde{\bar{q}}^{\bar{L}_0 - \frac{c}{24}} \right), \\ &= \sum_{j,\bar{\jmath},\alpha,\alpha'} \left(\Psi_x^{(j,\bar{\jmath};\alpha,\alpha')} \right)^* \Psi_y^{(j,\bar{\jmath};\alpha,\alpha')} \chi_j(\tilde{q}) \chi_{\bar{\jmath}}(\tilde{\bar{q}})\,. \end{aligned} \tag{2.21}$$

A second representation of the same partition function may be obtained by considering time running parallel to the defect lines. In this case, the definition of the disorder line (2.18) insures one may still construct two sets of generators L_n and $\bar{L}_n$ satisfying the Virasoro algebra (or more generally the chiral algebra $\mathcal{A}$). Hence the Hilbert space decomposes into irreducible representations,

$$\mathcal{H}_{x|y} = \bigoplus_{i,\bar{\imath}} V_{i\bar{\imath};x}{}^{y} \mathcal{R}_i \otimes \bar{\mathcal{R}}_{\bar{\imath}}\,, \tag{2.22}$$

for some non-negative integers $V_{i\bar{\imath};x}{}^{y}$, and the partition function becomes,

$$Z_{x|y} = \mathrm{Tr}_{\mathcal{H}_{x|y}} q^{L_0 - \frac{c}{24}} \bar{q}^{\bar{L}_0 - \frac{c}{24}} = \sum_{i,\bar{\imath}} V_{i\bar{\imath};x}{}^{y} \chi_i(q) \chi_{\bar{\imath}}(\bar{q})\,. \tag{2.23}$$

We may equate these two expressions using the modular transformation properties of the characters,

$$V_{i\bar{i};x}{}^{y} = \sum_{j,\bar{j},\alpha,\alpha'} S_{ji} S_{\bar{j}\bar{i}}\, \Psi_x^{(j,\bar{j};\alpha,\alpha')\,*} \Psi_y^{(j,\bar{j};\alpha,\alpha')} \,. \tag{2.24}$$

The task now is to solve this equation. A general discussion of this problem would take us too far a field[2] so here we simply state a class of solutions to (2.24) for theories with a charge conjugation modular invariant and refer the reader to the original paper of Zuber and Petkova [15]. Following [15] we observe using the Verlinde formula (2.3) that the ansatz,

$$\Psi_x^{(i,\bar{i})} = \frac{S_{xi}}{S_{0i}}\,, \tag{2.25}$$

satisfies (2.24) with,

$$V_{i\bar{i};x}{}^{y} = \sum_k N_{xi}{}^{k} N_{k\bar{i}}{}^{y} \,. \tag{2.26}$$

We call these defect lines *elementary*. Note that the defect lines are also in one-to-one correspondence with the primary fields and so we label them by the same set. Having obtained this solution, further solutions may be constructed by taking linear combinations with positive integer coefficients.

3. Considerations from TCSA

The truncated conformal space approach (TCSA) to studying perturbed quantum field theories is a very powerful tool. Initially proposed by Yurov and Zamolodchikov [17], the method has been successfully applied to boundary perturbations [18, 19, 20]. We will begin by reviewing the general formalism.

Consider a strip of width L with boundary condition a on the left, b on the right and a perturbation by relevant boundary fields $\phi_i(x) = \phi^{(bb)}_{i_\alpha;\mathbf{n}}(x)$ applied to the right boundary. We will study this deformation through the perturbed hamiltonian,

$$H_{\text{pert}} = H_0 + \sum_i \lambda_i \phi_i(L) \,. \tag{3.1}$$

Mapping this system to the upper half plane (UHP), the hamiltonian may be written in terms of operators acting on the UHP Hilbert space,

$$H_{ab}(L) = \frac{\pi}{L}\left(L_0 - \frac{c}{24}\right) + \sum_i \lambda_i \left(\frac{L}{\pi}\right)^{-h_i} \phi_i(1)\,, \tag{3.2}$$

where h_i is the conformal weight of the field ϕ_i. The idea behind the TCSA is to diagonalise $H_{ab}(L)$ on a finite dimensional subspace of the unperturbed Hilbert space. Such a truncation may be achieved by disregarding all states above a certain level, N. This produces an approximation to the finite size system, in particular to the spectrum and the matrix elements of local fields. The errors introduced by the cut-off are not particularly

[2]We refer the interested reader to [16, 8].

well understood, but by comparison with perturbation theory and exact results for integrable systems, this cut-off system can approximate the exact system very well for large ranges of L, with errors which reduce on increasing N.

The unperturbed Hilbert space for this system is,

$$\mathcal{H}_{ab} = \bigoplus_k {N_{ka}}^b \mathcal{R}_k \,. \tag{3.3}$$

So we denote a basis for the Hilbert space by $\{|\phi_{k_\gamma;\mathbf{n}}\rangle\}_{\mathbf{n}}$ where k labels the representation, $\mathbf{n}$ denotes the descendant states and the label $\gamma = 1, 2, \ldots, {N_{ka}}^b$ encodes any extra multiplicity there may be. We chose this basis to be such that the conformal generator L_0 is diagonal. To simplify notation, we often suppress indices labelling basis vectors.

The Hilbert space is equipped with a non-degenerate inner product. In conformal field theory this factorises into a contribution from the representation theory of $\mathcal{A}$ and a piece coming from the field theory,

$$G_{k,\ell} = \langle\phi_k|\phi_\ell\rangle = \langle\phi_{k_\alpha;\mathbf{n}}|\phi_{\ell_\beta;\mathbf{m}}\rangle = g(k)_{\mathbf{n},\mathbf{m}}\, \delta_{k\ell}\, \langle\phi_{k_\alpha;\mathbf{0}}|\phi_{k_\beta;\mathbf{0}}\rangle \,, \tag{3.4}$$

where the matrix $g(k)$ is determined entirely from representation theory. The matrix elements of the perturbed hamiltonian are as follows,

$$H|\phi_\ell\rangle = \sum_k H_{k\ell}|\phi_k\rangle \,, \qquad H_{k\ell} = \sum_j G^{-1}_{kj}\langle\phi_j|H|\phi_\ell\rangle \,. \tag{3.5}$$

There are two terms contributing to this. First the unperturbed hamiltonian: in our basis this acts diagonally,

$$\left[L_0 - \frac{c}{24}\right]_{k\ell} = \delta_{k\ell}\left(h_\ell - \frac{c}{24}\right) . \tag{3.6}$$

The interaction term is given in terms of the three point function:

$$[\phi_i(1)]_{k\ell} = \sum_j G^{-1}_{kj}\langle\phi_j|\phi_i(1)|\phi_\ell\rangle \,. \tag{3.7}$$

This can be calculated using the chiral algebra to reduce a general three point function to linear combination of a *finite* set of elementary three point functions multiplied by purely representation theoretic factors. Indeed, using the OPE (2.11) we obtain,

$$\langle\phi_j|\phi_i(1)|\phi_\ell\rangle = \sum_k \sum_{\gamma=1}^{{N_{i\ell}}^k} C_{i\ell;\gamma}^{\;\;k}\, \beta^\gamma\!\begin{bmatrix} k \\ i\,\ell \end{bmatrix} G_{jk} \,. \tag{3.8}$$

Placing this into (3.7) we find the two point functions cancel and so the perturbation matrices are given by representation theoretic data multiplied by a structure constant. Replacing all the indices gives the following formula for the perturbed hamiltonian,

$$[H_{ab}(L)]_{(k_\alpha,\mathbf{n}),(\ell_\beta,\mathbf{m})} = \delta_{(k_\alpha,\mathbf{n}),(\ell_\beta,\mathbf{m})}\frac{\pi}{L}\left(h_{\ell;\mathbf{m}} - \frac{c}{24}\right) + \sum_{i,\mathbf{p},\delta,\gamma} \lambda^{(bb)}_{i_\delta;\mathbf{p}} \left(\frac{L}{\pi}\right)^{-h_{i;\mathbf{p}}} C^{(bba)k_\alpha}_{i_\delta\ell_\beta;\gamma}\, \beta^\gamma\!\begin{bmatrix} & k;\mathbf{n} \\ i;\mathbf{p} & \ell;\mathbf{m} \end{bmatrix} . \tag{3.9}$$

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4. Disorder lines and boundary states

An important property of the disorder lines is their relationship with boundary conditions. Consider a boundary without field insertions and a disorder line running parallel to it. Looking at the action of a disorder line on a boundary state we use the Verlinde formula (2.3) to obtain,

$$\begin{aligned} X_x|a\rangle &= \left(\sum_j \frac{S_{xj}}{S_{0j}} P^{(j,\bar{j})}\right)\left(\sum_i \frac{S_{ai}}{\sqrt{S_{0i}}}|i\rangle\rangle\right) = \sum_j \frac{S_{xj}}{S_{0j}}\frac{S_{aj}}{\sqrt{S_{0j}}}|j\rangle\rangle \\ &= \sum_{b,j} N_{xa}{}^b \frac{S_{bj}}{\sqrt{S_{0j}}}|j\rangle\rangle = \sum_b N_{xa}{}^b |b\rangle \equiv |x \times a\rangle\,. \end{aligned} \tag{4.1}$$

And so for example,

$$|a\rangle = X_a|0\rangle\,, \tag{4.2}$$

where here $|0\rangle$ denotes the boundary state associated to the identity representation.

The partition function of the strip with Cardy boundary condition "a" on the left and "b" on the right is given in terms of the Verlinde fusion numbers as follows,

$$Z_{ab} = \sum_i N_{ia}{}^b \chi_i(q)\,. \tag{4.3}$$

In particular, one observes that,

$$Z_{0,a\times b} = Z_{a^\vee \times b^\vee,0} = Z_{a^\vee,b}\,. \tag{4.4}$$

These relations become more obvious when one considers a disorder line being pulled off one boundary, moved across the strip and fused onto the other. The question we would now like to address is: in what way are the boundary conditions a and b related to $a \times b$?

4.1 Embedding theorem

Consider a boundary with boundary condition b, a boundary field $\phi_i^{(bb)}(x_i) = \phi_{i_\alpha;\mathbf{n}}^{(bb)}(x_i)$ and a defect line X_a running parallel to it. The following manipulations are illustrated in figure 1. Applying the defect line to the boundary we see that away from the insertion points, the boundary condition is changed, $b \to c = a \times b$. Applied to the boundary field, the defect line cannot change the representation theoretic qualities of the field, but it may change other attributes. Hence we can write,

$$X_a\left[\phi_i^{(bb)}(x)\right] = \phi_{\rho(i)}^{(a\times b,a\times b)}(x) = \sum_n D_{in}\phi_n^{(a\times b,a\times b)}(x)\,. \tag{4.5}$$

which defines the linear map ρ and the numbers D_{in}. Now consider a pair of boundary fields. Before applying the defect line, we can use the OPE to simplify the product of the two fields,

$$\begin{aligned} X_a\left[\phi_i^{(bb)}(x_i)\phi_j^{(bb)}(x_j)\right] &= \sum_k C_{ij}^{(bbb)k}\,(x_i - x_j)^{-h_i-h_j+h_k} X_a\left[\phi_k^{(bb)}(x_j)\right] \\ &= \sum_{k,n} C_{ij}^{(bbb)k}\,(x_i - x_j)^{-h_i-h_j+h_k} D_{kn}\phi_n^{(a\times b,a\times b)}(x_j)\,, \end{aligned} \tag{4.6}$$

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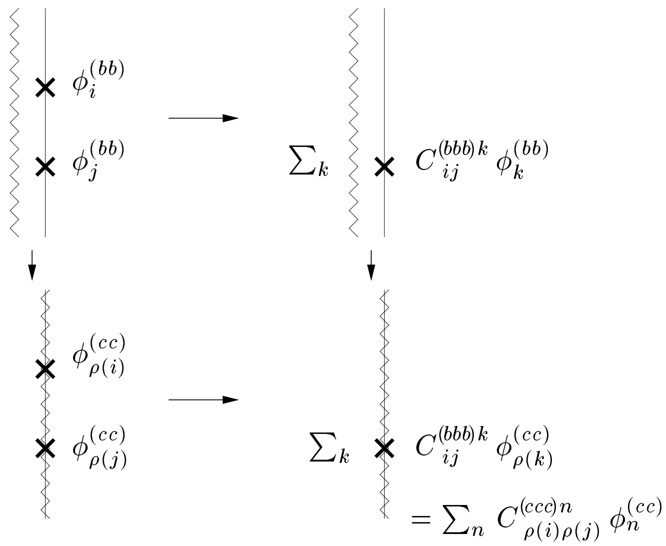

Figure 1: An illustration of the defect line manoeuvres used to obtain the embedding theorem.

wherein, to lighten notation we have absorbed the factors of β from the OPE (2.11) into a redefinition of the structure constants. Alternatively, we can apply the defect line before calculating the OPE. Let $c = a \times b$ then,

$$\begin{aligned} X_a\left[\phi_i^{(bb)}(x_i)\phi_j^{(bb)}(x_j)\right] &= \sum_{n,m} D_{in}D_{jm}\phi_n^{(cc)}(x_i)\phi_m^{(cc)}(x_j) \\ &= \sum_{n,m,p} D_{in}D_{jm}\,C_{nm}^{(ccc)p}\,(x_i-x_j)^{-h_i-h_j+h_p}\phi_p^{(cc)}(x_j)\,. \end{aligned} \tag{4.7}$$

It should not matter in which order we apply the defect line or take the OPE, thus we may identify (4.6) and (4.7) and obtain a homomorphism of algebras,

$$\sum_k C_{ij}^{(bbb)k}\,D_{kp} = \sum_{n,m} D_{in}D_{jm}\,C_{nm}^{(ccc)p}\ . \tag{4.8}$$

Furthermore, from the explicit form of the defect line operator (2.19) and (2.25) we see that applying the defect line to the identity operator on the b boundary gives a non-zero result,

$$X_a\left[\mathbb{1}^{(bb)}\right] \neq 0\,, \tag{4.9}$$

and hence the mapping X_a is one-to-one and has a (left) inverse. Indeed, assume there exists a field $\phi_i^{(bb)}(x)$ such that $X_a[\phi_i^{(bb)}(x)] = 0$, then consider,

$$X_a[\phi_i^{(bb)}(x)\phi_{i^\vee}^{(bb)}(y)] = X_a[\phi_i^{(bb)}(x)]X_a[\phi_{i^\vee}^{(bb)}(y)] = X_a[(x-y)^{-2h_i}\mathbb{1}^{(bb)}+\cdots] \neq 0\,, \tag{4.10}$$

giving a contradiction. Hence we have demonstrated an embedding of the operator algebra of the boundary b into that of the boundary $a \times b$. Let us formulate these observations in the following theorem:

let $c = a \times b$ be a relation between labels of boundary conditions. Then there exists an isomorphic embedding $b \subset c$ of the operator algebra of the b boundary condition into that of the c boundary condition.

4.2 Application to flows

We now turn our attention to perturbations of the boundary. Because the operator algebra of the boundary b can be realised within the boundary $c = a \times b$, it is natural to ask if the boundary flows of the boundary b also have a counterpart within the c boundary. This is indeed the case and in this section we will prove the following theorem:

let $c = a \times b$ be a relation between labels of boundary conditions and let there exist a boundary renormalisation group flow from boundary condition b to d. Then there also exists a flow from $a \times b$ to $a \times d$.

Moreover, through our investigations we will obtain further insight into the embedding theorem.

We begin our proof by considering the matrix elements of the perturbing hamiltonian on a pair of strips. We take strip A to be $\mathcal{H}_{a^\vee,b}$ and strip B to be $\mathcal{H}_{0,a\times b}$. Note that the Hilbert spaces of these two systems are identical,

$$\mathcal{H}_{a^\vee,b} = \bigoplus_k N_{ka^\vee}{}^b \mathcal{R}_k = \bigoplus_k N_{ab}{}^k \mathcal{R}_k = \bigoplus_{k,n} N_{ab}{}^n N_{k0}{}^n \mathcal{R}_k = \mathcal{H}_{0,a\times b} \tag{4.11}$$

and so the unperturbed hamiltonians are the same. Here we have used some of the properties of the fusion numbers collected in section 2. Let us label the vectors of the representation space $\mathcal{R}_k$ by $|\phi_{k_\gamma;\mathbf{n}}\rangle$ with $\gamma = 1, 2, \ldots, N_{ka^\vee}{}^b$. We will now show that for any perturbation on strip A, there exists a perturbation of strip B which has the same perturbing hamiltonian. That being so, by hypothesis we know there exists a flow taking,

$$\mathcal{H}_{a^\vee,b} \to \mathcal{H}_{a^\vee,d}\,. \tag{4.12}$$

Interpreting this as a flow on strip B we have,

$$\mathcal{H}_{0,a\times b} \to \mathcal{H}_{0,a\times d}\,, \tag{4.13}$$

as required.

We now show that we can find an appropriate hamiltonian on strip B. From the embedding theorem, we know that there exists a one-to-one mapping of each field on the right of strip A into the fields on the right of strip B preserving the chiral representation properties of that field. We also know from section 3 that the perturbed parts of the hamiltonians are given by a coupling constant times a structure constant and a representation theoretic matrix,

$$\left[H^A_{\mathrm{Pert}}\right]_{(k_\alpha,\mathbf{n}),(\ell_\beta,\mathbf{m})} = \sum_{i,\mathbf{p},\delta,\gamma} \lambda^{(bb)}_{i_\delta;\mathbf{p}} \left(\frac{L}{\pi}\right)^{-h_{i;\mathbf{p}}} C^{(bba^\vee)k_\alpha}_{i_\delta \ell_\beta;\gamma}\; \beta^\gamma\!\left[\begin{smallmatrix} & k;\mathbf{n} \\ i;\mathbf{p} & \ell;\mathbf{m}\end{smallmatrix}\right]. \tag{4.14}$$

To write the hamiltonian of strip B we need a little more notation. Note that the sectors of the Hilbert space and the Cardy boundary conditions on the right of the strip are labelled by the same set,

$$a \times b = \sum_k N_{ab}{}^k k = \sum_k N_{ka^\vee}{}^b k\,, \tag{4.15}$$

Indeed, a boundary field that changes the boundary condition ℓ_β into k_α also maps the Hilbert space sector ℓ_β into k_α ($\alpha = 1, 2, \ldots, N_{ab}{}^k$ and $\beta = 1, 2, \ldots, N_{ab}{}^\ell$) . Hence we can write,

$$\left[H^B_{\mathrm{Pert}}\right]_{(k_\alpha,\mathbf{n}),(\ell_\beta,\mathbf{m})} = \sum_{i,\mathbf{p},\rho,\gamma} \lambda^{(k_\alpha\ell_\beta)}_{i_\rho;\mathbf{p}} \left(\frac{L}{\pi}\right)^{-h_{i;\mathbf{p}}} C^{(k_\alpha\ell_\beta 0)k_\alpha}_{i_\rho\ell_\beta;\gamma} \;\beta^\gamma\!\begin{bmatrix} k;\mathbf{n} \\ i;\mathbf{p}\ \ell;\mathbf{m}\end{bmatrix} \tag{4.16}$$

Moreover, using (2.16)[3] we find that the structure constant is not equal to one if and only if the matrix $\beta^\gamma\!\left[{}^{\;k}_{i\ \ell}\right]$ vanishes: this observation implies that if we set,

$$\lambda^{(k_\alpha\ell_\beta)}_{i_\gamma;\mathbf{p}} = \sum_\delta \lambda^{(bb)}_{i_\delta;\mathbf{p}}\, C^{(bba^\vee)k_\alpha}_{i_\delta\ell_\beta;\gamma} \ , \tag{4.17}$$

we obtain the required identification between (4.14) and (4.16).

Actually to obtain (4.17) and prove our main result, we have not used the embedding theorem at all. Instead, equation (4.17) provides a candidate formula for the matrix D_{in} defined in equation (4.5): let k_α, ℓ_β label the elementary boundaries comprising $c = a \times b$ as above then,

$$X_a\left[\phi^{(bb)}_{i_\delta;\mathbf{p}}\right] = \sum_{k_\alpha,\ell_\beta,\gamma} C^{(bba^\vee)k_\alpha}_{i_\delta\ell_\beta;\gamma}\ \phi^{(k_\alpha\ell_\beta)}_{i_\gamma;\mathbf{p}} \ , \tag{4.18}$$

and equation (4.8) can be seen by a straightforward application of the pentagon identity for the fusing matrix [14] (also [13]). We have not shown that the embedding (4.18) is the same as that obtained using a defect line (4.5), however it seems very likely. One way to show this would be to use the 3-dimensional topological realisation of defect lines and boundary conditions introduced in [8].

5. Applications

In this section, we apply our rule to a number of renormalisation group flows studied in the literature.

5.1 Minimal model flows

In this subsection we consider the A-series unitary minimal models $M(m, m{+}1)$ with central charge,

$$c = 1 - \frac{6}{m(m+1)} \ . \tag{5.1}$$

The set labelling representations, primary bulk fields, boundary conditions and defect lines is the Kac table K where,

$$\begin{aligned} K' &= \{(r,s)\ :\ 1 \le r \le m{-}1\,,\ \ 1 \le s \le m\}\,, \\ K &= K'/\sim \qquad \text{where} \quad (r,s) \sim (m{-}r, m{+}1{-}s)\,. \end{aligned} \tag{5.2}$$

[3]Note that the multiplicity labels on k (and ℓ) in (4.16) range over $1, 2, \ldots, N_{kb^\vee}{}^c$ and so should not be identified with the multiplicity labels of (2.16). Instead, one should take (for example) $[\ell_\beta]_\varepsilon$ with $\varepsilon = 1, 2, \ldots, N_{\ell_\beta 0}{}^{\ell_\beta} = 1$. Hence we have suppressed these indices in (4.16).

Let us now consider the boundary renormalisation group flows in these models. In [21], Recknagel el. al. studied boundary flows in the unitary minimal models using perturbation theory. There it was found that on perturbing the boundary condition (r, s) by the field ϕ_{13} in the limit of large m, one flows to the superposition of boundary conditions,

$$(r, s) + \phi_{13} \to \oplus_{i=1}^{\min\{r,s\}} (|r-s|+2i-1, 1)\,, \tag{5.3}$$

Here we will extend this result to finite m and prove this flow exists for $s \leq \frac{m+1}{2}$. Our result depends on the work of [22] showing that the following flow exists,

$$(1, s) + \phi_{13} \to (s, 1)\,, \qquad s \leq \frac{m+1}{2}\,. \tag{5.4}$$

Indeed, by using our theorem we may identify the flows in $(1, s)$ with a subset of the flows in $(r, 1) \times (1, s) \simeq (r, s)$ as follows,

$$(r, s) \simeq (r, 1) \times (1, s) \;\to\; (r, 1) \times (s, 1) \simeq \oplus_{i=1}^{\min\{r,s,m-r,m-s\}} (|r-s|+2i-1, 1)\,. \tag{5.5}$$

To study the region $s > \frac{m+1}{2}$, we can use the identification in the minimal model $(r, s) \sim (m-r, m+1-s)$ to find [22],

$$(1, s) + \phi_{13} \to (s-1, 1)\,, \qquad s > \frac{m+1}{2}\,. \tag{5.6}$$

hence,

$$(r, s) \simeq (r, 1) \times (1, s) \;\to\; (r, 1) \times (s-1, 1) \simeq \oplus_{i=1}^{\min\{r,s-1,m-r,m+1-s\}} (|r-s+1|+2i-1, 1)\,. \tag{5.7}$$

It is generally believed that the flows (5.5) and (5.7) exist for all (r, s) with $1 < s < m$, and are generated by the same perturbation but with opposite sign of the coupling constant. This is borne out by numerical studies [23] (see [21]), TBA studies in the case of $r = 1$ [22] and is in agreement with the structure of boundary weights in lattice models [24, 25].

In [26] one of us studied perturbations of superpositions of boundary conditions of the form,

$$\omega = \oplus_{i=1}^{n} (r, s + 2i - 2)\,. \tag{5.8}$$

Note that this superposition may be written,

$$\omega \simeq (r, s+n-1) \times (1, n) \simeq (r, n) \times (1, s+n-1)\,. \tag{5.9}$$

Applying our theorem we reproduce and generalise the flows found in [26],

$$\omega \to \begin{cases} (r, s+n-1) \times (n, 1) \simeq \oplus_{i=1}^{\min\{r,n,m-r,m-n\}} (|r-n|+2i-1, s+n-1)\,, \\ (r, n) \times (s+n-1, 1) \simeq \oplus_{i=1}^{\min\{r,s+n-1,m-r,m-s-n+1\}} (|r-s-n+1|+2i-1, n)\,. \end{cases} \tag{5.10}$$

Furthermore, we can use the non-perturbative flow (5.6) to obtain,

$$\omega \to \begin{cases} (r, s+n-1) \times (n-1, 1) \simeq \oplus_{i=1}^{\min\{r,n-1,m-r,m-n+1\}} (|r-n+1|+2i-1, s+n-1)\,, \\ (r, n) \times (s+n-2, 1) \simeq \oplus_{i=1}^{\min\{r,s+n-2,m-r,m-s-n+2\}} (|r-s-n+2|+2i-1, n)\,. \end{cases} \tag{5.11}$$

These have also been observed using the absorption of boundary spin conjecture of [1, 2].

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5.2 Coset models

As a second application of this analysis, we will demonstrate that the "absorption of boundary spin" proposal of Fredenhagen and Schomerus [1, 2] may be reduced to a statement about more elementary flows in some cases. First we remind the reader of the proposal of Fredenhagen and Schomerus. The proposal concerns boundary flows in diagonal WZW coset models whose field content is specified by two affine Lie algebras $\mathfrak{h}_{k'} \subset \mathfrak{g}_k$. A set of elementary boundary conditions for this theory may be labelled by pairs of integrable representations of these algebras, (L, L') where L is an irreducible representation of $\mathfrak{g}_k$ and L' is an irreducible representation of $\mathfrak{h}_{k'}$. However, not all pairs allowed and some pairs label the same boundary condition.

In most cases, the selection and identification rules are specified in terms of two objects[4]: the *identification group* G_{id}, a subgroup of the set of simple currents in $\mathfrak{g}_k \otimes \mathfrak{h}_{k'}$ and which we assume to act fixed point free; and the *monodromy charge* associated to a simple current J, $Q_J(L) = h_J + h_L - h_{J\times L} \bmod \mathbb{Z}$, where h_L is the conformal weight of the representation L in either $\mathfrak{g}_k$ or $\mathfrak{h}_{k'}$ as appropriate.

- A pair (L, L') is allowed if $Q_J(L) = Q_{J'}(L')$ for all $(J, J') \in G_{\mathrm{id}}$.
- The pairs (L, L') and $(J{\times}L, J'{\times}L')$ are identified for each $(J, J') \in G_{\mathrm{id}}$.

A general boundary condition in the coset theory is a superposition of elementary boundary conditions so one may also associate a boundary condition to a pair of representations (L, L') by expanding in terms of irreducibles. We may now state the absorption of boundary spin conjecture:

let $(S, 0)$ and (L, L') be representations in $\mathfrak{g}_k \otimes \mathfrak{h}_{k'}$ such that $Q_J(S) + Q_J(L) = Q_{J'}(L')$ for all $(J, J') \in G_{id}$, then a flow exists between the following pair of boundary conditions,

$$(L, S^{\vee}|_{\mathfrak{h}} \times L') \to (L \times S, L')\,, \tag{5.12}$$

where the restriction on the left hand side involves taking the corresponding representation of the finite dimensional Lie algebra $\mathfrak{g}$, and restricting this to the subalgebra $\mathfrak{h} \subset \mathfrak{g}$. Then $S^{\vee}|_{\mathfrak{h}}$ is the representation of $\mathfrak{h}_{k'}$ associated to this restricted finite dimensional representation.

To use our result, we need to factorise (5.12) into pairs of representations allowed by the coset selection rules. This can be achieved if $Q_J(S) = 0$, in which case we use the fusion rules of the coset theory to rewrite (5.12) as follows,

$$(L, L') \times (0, S^{\vee}|_{\mathfrak{h}}) \to (L, L') \times (S, 0)\,. \tag{5.13}$$

In this form it is trivial to see using the machinery developed in this paper that the existence of the flow,

$$(0, S^{\vee}|_{\mathfrak{h}}) \to (S, 0)\,, \tag{5.14}$$

implies the existence of the others. Thus we have reduced a particular case of the absorption of boundary spin proposal of Fredenhagen and Schomerus to a statement about simpler flows. It remains to be shown that these elementary flows actually exist.

[4] We refer the reader to [2, 27] for details. Cosets can be defined in more general situations (see for example [28]), however like [1, 2], we will not consider such models.

6. Conclusions

In this note we have used the technology of defect lines to demonstrate an embedding of the operator algebra of a Cardy boundary condition b into that of $a \times b$. We then used this observation to show that if there exists a flow between boundary conditions b and d then there also exists a flow between $a \times b$ and $a \times d$ where $\times$ denotes the fusion product. Finally we considered applications of this rule to the Virasoro minimal models and to the absorption of boundary spin conjecture of Fredenhagen and Schomerus [1, 2].

One obvious question is in what way does the rule extend to more general non-diagonal rational conformal field theories? In this case the interrelationship between the various positive integer valued matrices discussed in [9, 16] will be relevant. A formalism in which both defect lines and boundary conditions are particularly transparent is the 3-dimensional topological quantum field theory framework championed in [8, 29]. Here for example, the action of (4.1) corresponds to the natural tensor product of the left Frobenius module corresponding to the boundary state with the bimodule corresponding to the defect line.

In arguing our theorem, we used the matrix elements of the perturbed hamiltonian in the unperturbed basis. We avoided questions of renormalisation by assuming these problems were solved. It would be nice to have a better understanding of the renormalisation procedure using level truncation. Such knowledge may lead to a new rigorous analytic method for studying renormalisation group flows.

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